ON THE EQUIVARIANT K-THEORY OF COMPACT SIMPLY-CONNECTED LIE GROUPS

CHI-KWONG FOK

ABSTRACT. In this note we present a proof of the result that, for any simply-connected, connected and compact Lie group G, viewed as a G-space with conjugation action, $K_G^*(G) \cong \Omega^*_{R(G)/\mathbb{Z}}$, the ring of Grothendieck differentials of the complex representation ring. The proof hinges on the computation of a certain equivariant index, which is a slight modification of the argument in [At].

1. Introduction

In [BZ], Brylinski and Zhang showed the following

Theorem 1.1. *For any compact connected Lie group G with $\pi_1(G)$ torsion-free, which is viewed as a G-space with conjugation action, $K_G^*(G)$ is isomorphic, as $R(G)$-algebras, to $\Omega^*_{R(G)/\mathbb{Z}}$, the ring of Grothendieck differentials of the representation ring $R(G)$.*

Their proof is outlined as follows.

(1) Let V be the underlying vector space of the representation ρ. The ring homomorphism $\varphi : \Omega^*_{R(G)/\mathbb{Z}} \to K_G^*(G)$ defined by

$$\varphi(\rho) = [G \times V] \in K_G^0(G)$$

$$\varphi(d\rho) = [0 \longrightarrow G \times \mathbb{R} \times V \xrightarrow{\psi} G \times \mathbb{R} \times V \longrightarrow 0] \in K_G^{-1}(G)$$

where $\psi(g, t, v) = (g, t, t\rho(g)v)$ and the G-action is the diagonal one, is an $R(G)$-algebra homomorphism.

(2) By using Hodgkin's spectral sequence for equivariant K-theory ([HS]) and Segal's localization theorem for equivariant K-theory ([S]), they showed that $K_G^*(G)$ is a free $R(G)$-module of rank 2^l, where $l = \mathrm{rank}(G)$.

(3) Consider the composition of maps

$$\Omega^*_{R(G)/\mathbb{Z}} \xrightarrow{\varphi} K_G^*(G) \xrightarrow{\alpha} K_T^*(G) \xrightarrow{i^*} K_T^*(T) \cong \Omega^*_{R(T)/\mathbb{Z}}$$

If

$$i^* \circ \alpha \circ \varphi : \Omega^*_{R(G)/\mathbb{Z}} \xrightarrow{\cong} (\Omega^*_{R(T)/\mathbb{Z}})^W \tag{1}$$

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then φ is injective. Moreover, α is injective by the 'splitting principle', and i^*, being a localization map, is injective on $R(T)$-torsion-free elements, in particular $\mathrm{Im}(\alpha)$, by Segal's localization theorem. Thus $i^* \circ \alpha$ is injective. Noting that $\mathrm{Im}(i^* \circ \alpha) \subseteq K_T^*(T)^W$ and assuming that the isomorphism (1) is true, one can conclude that φ is surjective and hence an isomorphism. A significant part of their proof is devoted to showing that (1) holds, using algebro-geometric arguments.

In this note, we present an alternative proof of Theorem 1.1 for the special case of simply-connected compact Lie groups. We replace step (3) of their proof with an index theory argument which is essentially a slight modification of the argument in Atiyah's computation of ordinary K-theory of simply-connected compact Lie groups ([At]), followed by an abelianization result in equivariant K-theory ([HLS]). Throughout this note G denotes any simply-connected, connected and compact Lie group and T its maximal torus.

2. The proof

Lemma 2.1. *$K_T^*(G)$ is a free $R(T)$-module of rank 2^l.*

Proof. By Lemma 2.5 of [BZ], $K_T^*(G) \cong K_G^*(G) \otimes_{R(G)} R(T)$ as $R(T)$-modules. Since $K_G^*(G)$ is a free $R(G)$-module of rank 2^l (Lemmata 4.4 and 4.5 of [BZ]), the result follows. □

Definition 2.2. Let $\mathrm{Ind}_T^G : K_T^*(G) \to K_T^*(\mathrm{pt})$ be the composition of Thom isomorphism $\mathrm{th}_{TG} : K_T^*(G) \to K_T^{*+\dim G}(TG)$ and the T-equivariant index map $K_T^*(TG) \to K_T^*(\mathrm{pt})$.

Lemma 2.3. $Ind_T^G(u) = \dfrac{Ind_T^T i^* u}{\prod_{\alpha \in \Delta^-}(1 - e^\alpha)}$

Proof. This is essentially the Atiyah-Segal localization formula for equivariant index ([AS]). Consider the following commutative diagram

$$\begin{array}{ccccc} TT & \xrightarrow{i_2} & TG|_T & \xrightarrow{i_1} & TG \\ & \searrow^{\pi} & \downarrow{\scriptstyle \pi_2} & & \downarrow{\scriptstyle \pi_1} \\ & & T & \xrightarrow{i} & G \end{array}$$

Then

$$\begin{aligned} \mathrm{Ind}_T^G(u) &= \mathrm{Ind}_T^{TG}(\pi_1^* u \cdot \tau_{TG/G}) \\ &= \mathrm{Ind}_T^{TT} \frac{i_2^* \circ i_1^*(\pi_1^* u \cdot \tau_{TG/G})}{\pi^*(\bigoplus_i (-1)^i \bigwedge^i N \otimes_{\mathbb{R}} \mathbb{C})} \end{aligned}$$

where N is the normal bundle of T in G. Note that $N \cong T \times \mathfrak{g}/\mathfrak{t} = T \times \bigoplus_{\alpha \in \Delta^+} \mathfrak{m}_\alpha$. So

$$\bigoplus_i (-1)^i \bigwedge^i N \otimes_{\mathbb{R}} \mathbb{C} = \prod_{\alpha \in \Delta} (1 - e^\alpha) \otimes 1 \in R(T) \otimes K^*(T) \cong K_T^*(T)$$

$$\begin{aligned} i_2^* \circ i_1^*(\pi_1^* u \cdot \tau_{TG/G}) &= i_2^*(\pi_2^*(i^*u) \cdot \tau_{TG_{|T}/T}) \\ &= i_2^*(\pi_2^*(i^*u) \cdot (\pi_3^* \tau_{TT/T} \cdot \tau_{TG_{|T}/TT})) \\ &= \pi^*(i^*u) \cdot \tau_{TT/T} \cdot i_2^* \tau_{TG_{|T}/TT} \\ &= \pi^*(i^*u) \cdot \tau_{TT/T} \cdot (\bigoplus_i (-1)^i \pi^*(\bigwedge^i N)) \\ &= \pi^*(i^*u \cdot (\prod_{\alpha\in\Delta^+}(1-e^\alpha)\otimes 1)) \cdot \tau_{TT/T} \end{aligned}$$

It follows that

$$\begin{aligned} &\mathrm{Ind}_T^{TT} \frac{i_2^* \circ i_1^*(\pi_1^* u \cdot \tau_{TG/G})}{\pi^*(\bigoplus_i (-1)^i \bigwedge^i N \otimes_{\mathbb{R}} \mathbb{C})} \\ =&\mathrm{Ind}_T^{TT} \pi^* \left(\frac{i^*u \cdot (\prod_{\alpha\in\Delta^+}(1-e^\alpha \otimes 1))}{\prod_{\alpha\in\Delta}(1-e^\alpha)\otimes 1} \right) \cdot \tau_{TT/T} \\ =&\mathrm{Ind}_T^T \frac{i^*u}{\prod_{\alpha\in\Delta^-}(1-e^\alpha)\otimes 1} \\ =&\frac{\mathrm{Ind}_T^T i^*u}{\prod_{\alpha\in\Delta^-}(1-e^\alpha)} \end{aligned}$$

□

Definition 2.4. $\beta(\rho) := \varphi(d\rho)$

Proposition 2.5. $i^*\beta(\rho) = \sum_{j=1}^{dim\rho} e^{\lambda_j} \otimes \beta(\lambda_j) \in K_T^{-1}(T)$, *where* λ_j *are the weights of* ρ.

Proof. This follows easily by restricting the complex of T-equivariant vector bundles

$$0 \longrightarrow G \times \mathbb{R} \times V \xrightarrow{\psi} G \times \mathbb{R} \times V \longrightarrow 0$$

to

$$0 \longrightarrow T \times \mathbb{R} \times V \xrightarrow{\psi|_{T\times\mathbb{R}\times V}} T \times \mathbb{R} \times V \longrightarrow 0$$

and noting that the second complex can be decomposed into a direct sum of complexes of 1-dimensional T-equivariant vector bundles, each of which corresponds to a weight of ρ. □

The following is an equivariant analogue of Lemma 3 of [At].

Lemma 2.6.

$$i^* \left(\prod_{i=1}^{l} \beta(\rho_i) \right) = d_G \otimes \prod_{i=1}^{l} \beta(\varpi_i)$$

where ρ_i *is the* i*-th fundamental representation,* ϖ_i *is the* i*-th fundamental weight and* $d_G \in R(T)$ *is the Weyl denominator.*

Proof. By Proposition 2.5,

$$i^*\left(\prod_{i=1}^{l}\beta(\rho_i)\right)=\prod_{i=1}^{l}\sum_{j=1}^{\dim\rho_i}e^{\varpi_{ij}}\otimes\beta(\varpi_{ij}) \tag{2}$$

The right-hand side must be of the form $A\otimes\prod_{i=1}^{l}\beta(\varpi_i)$ for some $A\in R(T)$. Since $i^*(\prod_{i=1}^{l}\beta(\rho_i))$ is W-invariant and $\prod_{i=1}^{l}\beta(\varpi_i)$ is anti-W-invariant, A is anti-W-invariant and of the form $\displaystyle\sum_{w\in W}\sum_{i=1}^{n}\mathrm{sgn}(w)e^{w\cdot\gamma_i}$, where γ_i lie in the positive Weyl chamber. Let γ_1 be the highest weight among $\lambda_1,\cdots,\lambda_n$. By comparing the right-hand side of Eq. (2) and $A\otimes\prod_{i=1}^{l}\beta(\varpi_i)$, $\lambda_1=\displaystyle\sum_{i=1}^{l}\varpi_i:=\theta$ (the half sum of positive roots). Note that λ_1 is the lowest weight in the interior of the positive Weyl chamber, and that if γ lies on any wall of the chamber, $\displaystyle\sum_{w\in W}\mathrm{sgn}(w)e^{w\cdot\gamma}=0$. As a result, $A=\displaystyle\sum_{w\in W}\mathrm{sgn}(w)e^{w\cdot\theta}=d_G$. □

Lemma 2.7. *The map* $Ind^T:K^*(T)\stackrel{th_{TT/T}}{\longrightarrow}K^*(TT)\stackrel{Ind^{TT}}{\longrightarrow}K^*(pt)$ *sends* $\displaystyle\prod_{i=1}^{l}\beta(\varpi_i)$ *to 1.*

Proof. See [At]. □

Corollary 2.8. $Ind_T^G\left(\displaystyle\prod_{i=1}^{l}\beta(\rho_i)\right)=e^{-\theta}$

Proof.

$$\begin{aligned}\mathrm{Ind}_T^G\left(\prod_{i=1}^{l}\beta(\rho_i)\right)&=\frac{\mathrm{Ind}_T^T\left(\sum_{w\in W}\mathrm{sgn}(w)e^{w\cdot\theta}\otimes\prod_{i=1}^{l}\beta(\varpi_i)\right)}{\prod_{\alpha\in\Delta^-}(1-e^{\alpha})}\\&=\frac{\prod_{\alpha\in\Delta^+}(e^{\frac{\alpha}{2}}-e^{-\frac{\alpha}{2}})\otimes\mathrm{Ind}^T\prod_{i=1}^{l}\beta(\varpi_i)}{\prod_{\alpha\in\Delta^+}(1-e^{-\alpha})}\\&=e^{-\theta}\end{aligned}$$

□

Proposition 2.9. $\varphi:\bigwedge_{R(T)}(d\rho_1,\cdots,d\rho_l)\to K_T^*(G)$ *is an algebra isomorphism.*

Proof. Define $k:K_T^*(G)\to\mathrm{Hom}_{R(T)}(\bigwedge_{R(T)}(d\rho_1,\cdots,d\rho_l),R(T))$ by $x\mapsto(y\mapsto\mathrm{Ind}_T^T(x\varphi(y)))$. Corollary 2.8 implies that $k\circ\varphi$ is an $R(T)$-module isomorphism (as the equivariant index $e^{-\theta}$ is invertible in $R(T)$), which in turn implies that φ is injective, and $\mathrm{Im}(\varphi)$ is a direct summand of $K_T^*(G)$. Being both of rank 2^l, $\mathrm{Im}(\varphi)=K_T^*(G)$. □

Theorem 2.10. *(1) $\alpha : K^*_G(G) \to K^*_T(G)^W$ is a ring isomorphism. Here W acts on the coefficient ring $K^*_T(pt) = R(T)$ in the following sense. If E is a T-equivariant vector bundle, then $w \cdot E$ is the same vector bundle with a new T-action given by $t * v = (w \cdot t) \cdot v$.*

(2) $\varphi : \bigwedge_{R(T)}(d\rho_1, \cdots, d\rho_l) \to K^*_T(G)$ is W-equivariant.

(3) $\varphi : \bigwedge_{R(G)}(d\rho_1, \cdots, d\rho_l) \to K^*_G(G)$ is an algebra isomorphism.

Proof. (1) is a consequence of Lemma 2.1 and part (i) of Corollary 4.10 of [HLS]. (2) is obvious from the definition of φ. (3) follows immediately from Proposition 2.9, (1) and (2). □

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Department of Mathematics,
Cornell University,
Ithaca, NY 14853-4201,
USA
E-mail: ckfok@math.cornell.edu
URL: http://www.math.cornell.edu/∼ckfok